Curvature spectra of simple Lie groups

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Abstract The Killing form β of a real (or complex) semisimple Lie group G is a left-invariant pseudo-Riemannian (or, respectively, holomorphic) Einstein metric. Let Ω denote the multiple of its curvature operator, acting on symmetric 2-tensors, with the factor chosen so that $\Omega\beta = 2\beta$. The result of Meyberg [8], describing the spectrum of Ω in complex simple Lie groups G, easily implies that 1 is not an eigenvalue of Ω in any real or complex simple Lie group G except those locally isomorphic to $\mathrm{SU}(p,q)$, or $\mathrm{SL}(n,\mathrm{I\!R})$, or $\mathrm{SL}(n,\mathbb{C})$ or, for even n only, $\mathrm{SL}(n/2,\mathrm{I\!H})$, where $p \geq q \geq 0$ and $p+q=n>2$. Due to the last conclusion, on simple Lie groups G other the ones just listed, nonzero multiples of the Killing form β are isolated among left-invariant Einstein metrics. Meyberg's theorem also allows us to understand the kernel of Λ, which is another natural operator. This in turn leads to a proof of a known, yet unpublished, fact: namely, that a semisimple real or complex Lie algebra with no simple ideals of dimension 3 is essentially determined by its Cartan three-form.

Keywords Simple Lie group · indefinite Einstein metric · left-invariant Einstein metric · Cartan three-form

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1 Introduction

Every real Lie group G carries a distinguished left-invariant torsionfree connection D, defined by $D_x y = [x,y]/2$ for all left-invariant vector fields x and y. In view of the Jacobi identity, the curvature tensor of D is D-parallel, and hence so is the Ricci tensor of D, equal

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to a nonzero multiple of the Killing form β. Our convention about β reads

$$\beta(x,x) = \operatorname{tr}[\operatorname{Ad}x]^2 \quad \text{for any } x \text{ in the Lie algebra } \mathfrak{g} \text{ of } G. \tag{1.1}$$

Thus, if G is semisimple, β constitutes a bi-invariant, locally symmetric, non-Ricci-flat pseudo-Riemannian Einstein metric on G, with the Levi-Civita connection D. We denote by $\Omega : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$ a specific multiple of the curvature operator of the metric β, acting on symmetric symmetric bilinear forms $\sigma : \mathfrak{g} \to \mathfrak{g}$, so that, whenever $x, y \in \mathfrak{g}$,

$$\begin{aligned} &\text{a)}\ \ [\Omega\sigma](x,y) = 2\operatorname{tr}[(\operatorname{Ad}x)(\operatorname{Ad}y)\Sigma], \quad \text{for } \Sigma : \mathfrak{g} \to \mathfrak{g} \text{ with} \\ &\text{b)}\ \ \sigma(x,y) = \beta(\Sigma x, y). \end{aligned} \tag{1.2}$$

See Remark 2.4. The same formula (1.2) defines the operator Ω in a *complex* semisimple Lie group G, acting on symmetric complex-bilinear forms σ. We then identify Ω with the analogous curvature operator for the ($\mathbb{C}$-bilinear) Killing form β, treating the latter as a holomorphic Einstein metric on the underlying complex manifold of G.

The structure of Ω in complex simple Lie groups is known from the work of Meyberg [8], who showed that Ω is diagonalizable and described its spectrum. For the reader's convenience, we reproduce Meyberg's theorem in an appendix. His result easily leads to a similar description of the spectrum of Ω in real simple Lie algebras $\mathfrak{g}$, which we state as Theorem 4.1 and derive in Section 4 from the fact that, given any such $\mathfrak{g}$,

$$\begin{aligned} &\text{a)}\ \text{either } \mathfrak{g} \text{ is a real form of a complex simple Lie algebra } \mathfrak{h}, \text{ or} \\ &\text{b)}\ \mathfrak{g} \text{ arises by treating a complex simple Lie algebra } \mathfrak{h} \text{ as real.} \end{aligned} \tag{1.3}$$

See [6, Lemma 4 on p. 173]. The Lie-algebra isomorphism types of real simple Lie algebras $\mathfrak{g}$ thus form two disjoint classes, characterized by (1.3.a) and (1.3.b).

For both real and complex semisimple Lie groups G, studying Ω can be further motivated as follows. Let 'metrics' on G be, by definition, pseudo-Riemannian or, respectively, holomorphic, and $\mathscr{E}$ denote the set of Levi-Civita connections of left-invariant Einstein metrics on G. Then, as shown in [5, Remark 12.3], whenever a semisimple Lie group G has the property that 1 is *not* an eigenvalue of Ω, the Levi-Civita connection D of its Killing form β is an isolated point of $\mathscr{E}$. The converse implication holds except when G is locally isomorphic to $\mathrm{SU}(n)$, with $n \ge 3$. See [5, Theorems 22.2 and 22.3].

In a real/complex Lie algebra $\mathfrak{g}$, we define $\Lambda : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ by

$$(\Lambda\sigma)(x,y,z,z') = \sigma([x,y],[z,z']) + \sigma([y,z],[x,z']) + \sigma([z,x],[y,z']). \tag{1.4}$$

Thus, Λ is a real/complex-linear operator, sending symmetric bilinear forms σ on $\mathfrak{g}$ to exterior 4-forms on $\mathfrak{g}$. For the Killing form β one has $\beta([x,y],[z,z']) = \beta([[x,y],z],z')$, as $\operatorname{Ad}z$ is β-skew-adjoint. Furthermore, by the Jacobi identity and (1.1) – (1.2.a),

$$\text{i)}\ \ \Lambda\beta = 0, \qquad \text{ii)}\ \ \Omega\beta = 2\beta. \tag{1.5}$$

If, in addition, $\mathfrak{g}$ is semisimple, there is also the operator $\Pi : [\mathfrak{g}^*]^{\otimes 4} \to [\mathfrak{g}^*]^{\otimes 2}$ with

$$\Pi(\xi \otimes \xi' \otimes \eta \otimes \eta') = \beta([x,x'],\cdot) \otimes \beta([y,y'],\cdot), \tag{1.6}$$

for $\xi, \xi', \eta, \eta' \in \mathfrak{g}^*$, where $x, x', y, y' \in \mathfrak{g}$ are characterized by $\xi = \beta(x,\cdot)$, $\xi' = \beta(x',\cdot)$, $\eta = \beta(y,\cdot)$, $\eta' = \beta(y',\cdot)$. Formula (3.1) below shows that $\Pi([\mathfrak{g}^*]^{\wedge 4}) \subset [\mathfrak{g}^*]^{\odot 2}$.

Our first main result, established in Section 3, relates Ω to $\Pi\Lambda : [\mathfrak{g}^*]^{\odot 2} \to [\mathfrak{g}^*]^{\odot 2}$, the composite of Λ and the restriction of Π to the subspace $[\mathfrak{g}^*]^{\wedge 4} \subset [\mathfrak{g}^*]^{\otimes 4}$.

Theorem A *Let* Ω, Λ *and* Π *be the operators defined by* (1.2), (1.4) *and* (1.6) *for a given semisimple real/complex Lie algebra* $\mathfrak{g}$. *Then* $2\Pi\Lambda = -(\Omega + \mathrm{Id})(\Omega - 2\,\mathrm{Id})$.

Next, in Section 5, we use Meyberg's result and Theorem A to obtain the following description of $\operatorname{Ker}\Lambda$ for semisimple Lie algebras $\mathfrak{g}$. It provides a crucial step in our proof of Theorem C (see below).

Theorem B *Given a real/complex semisimple Lie algebra* $\mathfrak{g}$ *with a direct-sum decomposition* $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_s$ *into simple ideals,* $s \geq 1$, *let* Λ *and* Λ_i *denote the operator defined by* (1.4) *for* $\mathfrak{g}$ *and, respectively, its analog for the* i*th summand* $\mathfrak{g}_i$.

(i) $\operatorname{Ker}\Lambda = \operatorname{Ker}\Lambda_1 \oplus \ldots \oplus \operatorname{Ker}\Lambda_s$, *where* $[\mathfrak{g}_i^*]^{\odot 2} \subset [\mathfrak{g}^*]^{\odot 2}$ *via trivial extensions.*
(ii) $\Lambda = 0$ *if* $\dim\mathfrak{g} = 3$.
(iii) $\dim\operatorname{Ker}\Lambda = 12$ *if* $\mathfrak{g}$ *is simple and* $\dim\mathfrak{g} = 6$, *which happens only when* $\mathfrak{g}$ *is real and isomorphic to the underlying real Lie algebra of* $\mathfrak{sl}(2,\mathbb{C})$, *while* $\operatorname{Ker}\Lambda$ *then consists of the real parts of all symmetric* $\mathbb{C}$*-bilinear functions* $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$.
(iv) $\dim\operatorname{Ker}\Lambda \in \{1,2\}$ *whenever* $\mathfrak{g}$ *is simple and* $\dim\mathfrak{g} \notin \{3,6\}$, *while* $\operatorname{Ker}\Lambda$ *is then spanned either by the Killing form* β, *or by* $\operatorname{Re}\beta^{\mathfrak{h}}$ *and* $\operatorname{Im}\beta^{\mathfrak{h}}$. *The former case occurs if* $\mathfrak{g}$ *is complex, or real of type* (1.3.a), *the latter if* $\mathfrak{g}$ *is real of type* (1.3.b), *with* $\beta^{\mathfrak{h}}$ *denoting the Killing form of a complex simple Lie algebra* $\mathfrak{h}$ *in* (1.3.b).

Finally, one defines the *Cartan three-form* $C \in [\mathfrak{g}^*]^{\wedge 3}$ of a Lie algebra $\mathfrak{g}$ by

$$C = \beta([\cdot,\cdot],\cdot), \qquad \text{where } \beta \text{ denotes the Killing form.} \tag{1.7}$$

The following result has been known for decades, although no published proof of it seems to exist [3]. By an *isomorphism of the Cartan three-forms* we mean here a vector-space isomorphism of the Lie algebras in question, sending one three-form onto the other.

Theorem C *Let* $\mathfrak{g}$ *be a real/complex semisimple Lie algebra with a fixed direct-sum decomposition into simple ideals, which we briefly refer to as the "summands" of* $\mathfrak{g}$.

(i) *If* $\mathfrak{h}$ *is a real/complex Lie algebra, the Cartan three-forms of* $\mathfrak{g}$ *and* $\mathfrak{h}$ *are isomorphic and, in the real case,* $\mathfrak{g}$ *has no summands of dimension* 3, *then* $\mathfrak{h}$ *is isomorphic to* $\mathfrak{g}$.
(ii) *If* $\mathfrak{g}$ *contains no summands of dimension* 3 *or* 6, *then every automorphism of the Cartan three-form of* $\mathfrak{g}$ *is a Lie-algebra automorphism of* $\mathfrak{g}$ *followed by an operator that acts on each summand as the multiplication by a cubic root of* 1.
(iii) *If* $\mathfrak{g}$ *is the underlying real Lie algebra of a complex simple Lie algebra and* $\dim\mathfrak{g} \neq 6$, *then every automorphism of the Cartan three-form of* $\mathfrak{g}$ *is complex-linear or antilinear.*

Conversely, if $\mathfrak{g}$ *has* k *summands of dimension* 3 *and* l *summands of dimension* 6, *then the Lie-algebra automorphisms of* $\mathfrak{g}$ *form a subgroup of codimension* $5k + 12l$ *in the automorphism group of the Cartan three-form.*

We derive Theorem C from Theorem B, in Section 7.

2 Preliminaries

Suppose that $\mathfrak{g}$ is the underlying real Lie algebra of a complex Lie algebra $\mathfrak{h}$. We denote by β and C the Killing form and Cartan three-form of $\mathfrak{g}$, cf. (1.1) and (1.7), by Λ the

operator in (1.4) associated with $\mathfrak{g}$, and use the symbols $\beta^{\mathfrak{h}}, C^{\mathfrak{h}}, \Lambda^{\mathfrak{h}}$ for their counterparts corresponding to $\mathfrak{h}$. Obviously, whenever $\sigma : \mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is a symmetric $\mathbb{C}$-bilinear form,

$$\text{i) } \beta = 2\,\mathrm{Re}\,\beta^{\mathfrak{h}}, \quad \text{ii) } C = 2\,\mathrm{Re}\,C^{\mathfrak{h}}, \quad \text{iii) } \Lambda(\mathrm{Re}\,\sigma) = \mathrm{Re}\,(\Lambda^{\mathfrak{h}}\sigma). \tag{2.1}$$

For (2.1.i), see also [5, formula (13.1)]. With $\mathfrak{g}$ and $\mathfrak{h}$ as above, it is clear from (2.1.i) that

$$\begin{array}{l}\mathrm{Re}\,\beta^{\mathfrak{h}} \text{ and } \mathrm{Im}\,\beta^{\mathfrak{h}} \text{ span the real space of symmetric bilinear forms } \sigma \text{ on } \mathfrak{g} \text{ arising}\\ \text{via (1.2.b) from linear endomorphisms } \Sigma \text{ which are complex multiples of } \mathrm{Id}.\end{array} \tag{2.2}$$

Furthermore, (2.1.i) also implies, for dimensional reasons, that

$$\begin{array}{l}\text{the real parts of symmetric } \mathbb{C}\text{-bilinear functions } \mathfrak{g}\times\mathfrak{g}\to\mathbb{C} \text{ form the image}\\ \text{under (1.2.b) of the space of } \mathbb{C}\text{-linear } \beta^{\mathfrak{h}}\text{-self-adjoint endomorphisms of } \mathfrak{h},\end{array} \tag{2.3}$$

as the former space obviously contains the latter.

Let $\mathfrak{g}$ now be a Lie algebra over the scalar field $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$. A fixed basis of $\mathfrak{g}$ allows us to represent elements x,y of $\mathfrak{g}$, symmetric bilinear forms σ on $\mathfrak{g}$, and the Lie-algebra bracket operation $[\,,\,]$ by their components x^i, y^i, σ_{ij} and $C_{ij}{}^k$ (the structure constants of $\mathfrak{g}$), so that $\sigma(x,y) = \sigma_{ij}x^iy^j$ and $[x,y]^k = C_{ij}{}^kx^iy^j$. Repeated indices are summed over. The Cartan three-form C with (1.7) has the components $C_{ijk} = C_{ij}{}^r\beta_{kr}$, where β is the Killing form. The definition (1.1) of β, its bi-invariance, and the Jacobi identity now read

$$\begin{array}{ll}\text{i) } \beta_{ij} = C_{ip}{}^qC_{jq}{}^p, & \text{ii) } C_{ijk} \text{ is skew-symmetric in } i,j,k,\\ \text{iii) } C_{ij}{}^qC_{qk}{}^l + C_{jk}{}^qC_{qi}{}^l + C_{ki}{}^qC_{qj}{}^l = 0. & \end{array} \tag{2.4}$$

In the remainder of this section $\mathfrak{g}$ is also assumed to be semisimple. We can thus lower and raise indices using the components β_{ij} of the Killing form β and β^{ij} of its reciprocal: $C^k{}_p{}^q = \beta^{kr}C_{rp}{}^q$, and $C_j{}^{sp} = \beta^{sk}C_{jk}{}^p$. For any $x,y,z\in\mathfrak{g}$, one has $2\,\mathrm{tr}\,[(\mathrm{Ad}\,x)(\mathrm{Ad}\,y)(\mathrm{Ad}\,z)] = C(x,y,z)$, where C is the Cartan three-form given by (1.7). Equivalently,

$$2C_{ir}{}^pC_{jq}{}^rC_{kp}{}^q = C_{ijk}. \tag{2.5}$$

In fact, by successively using the equalities $C^k{}_p{}^q = C_p{}^{qk}$ and $C_i{}^{rp} = -C_i{}^{pr}$ (both due to (2.4.ii)), then again (2.4.ii), (2.4.iii), and (2.4.i–ii), we get $2C_{ir}{}^pC_{jq}{}^rC^k{}_p{}^q = 2C_i{}^{rp}C_{jqr}C_p{}^{qk} = C_i{}^{rp}(C_{jqr}C_p{}^{qk} - C_{jqp}C_r{}^{qk}) = C_i{}^{rp}(C_{jr}{}^qC_{qp}{}^k + C_{pj}{}^qC_{qr}{}^k) = -C_i{}^{rp}C_{rp}{}^qC_{qj}{}^k = \delta_i^qC_{qj}{}^k = C_{ij}{}^k$. Lowering the index k, we obtain (2.5). Next, we introduce the linear operator

$$T : [\mathfrak{g}^*]^{\otimes 2}\to[\mathfrak{g}^*]^{\otimes 2} \quad\text{with}\quad (T\sigma)_{ij} = T_{ij}^{kl}\sigma_{kl}, \quad\text{where}\quad T_{ij}^{kl} = 2C_{ip}{}^kC_j{}^{lp}. \tag{2.6}$$

Lemma 2.1 *For T and the operator $\Omega : [\mathfrak{g}^*]^{\odot 2}\to[\mathfrak{g}^*]^{\odot 2}$ given by* (1.2),

(a) *T leaves the subspaces $[\mathfrak{g}^*]^{\odot 2}$ and $[\mathfrak{g}^*]^{\wedge 2}$ invariant,*
(b) *Ω coincides with the restriction of T to $[\mathfrak{g}^*]^{\odot 2}$.*

Proof Our claim is obvious from (2.6) and the fact that, by (2.6), $T\sigma$ is the same as $\Omega\sigma$ in (1.2), except that now $\sigma : \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$ need not be symmetric. □

Lemma 2.2 *For any complex simple Lie algebra $\mathfrak{g}$ and $\Omega : [\mathfrak{g}^*]^{\odot 2}\to[\mathfrak{g}^*]^{\odot 2}$ with* (1.2),

(a) Ω *is diagonalizable,*
(b) 2 *is an eigenvalue of* Ω *with multiplicity* 1,
(c) 0 *is not an eigenvalue of* $\Omega^{\mathfrak{h}}$,
(d) Ω *has the eigenvalue* 1 *if and only if* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{sl}(n,\mathbb{C})$ *for some* $n \geq 3$.

Proof This is a special case of Meyberg's theorem, stated in the Appendix. □

Remark 2.3 The isomorphism types of all complex simple Lie algebras are: $\mathfrak{sl}_n$, for $n \geq 2$, $\mathfrak{sp}_n$ (even $n \geq 4$), $\mathfrak{so}_n$ with $n \geq 7$, as well as $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7$ and $\mathfrak{e}_8$. See [9, pp. 8 and 77].

Remark 2.4 The curvature operator of a (pseudo)Riemannian metric γ on a manifold, acting on symmetric 2-tensors, has been studied by various authors [4], [2], [1, pp. 51–52]. It is given by $\sigma \mapsto \tau$, where $2\tau_{ij} = \gamma^{pq} R_{ipj}{}^{k} \sigma_{qk}$ in terms of components relative to a basis of the tangent space at any point, the sign convention about the curvature tensor R being that a Euclidean tangent plane with an orthonormal basis x, y has the sectional curvature $\gamma_{pq} R_{ijk}{}^{p} x^i y^j x^k y^q$. When γ is the Killing form β of a semisimple Lie group G, treated as a left-invariant metric (see the lines following (1.1)), this operator equals $-\Omega/16$, for Ω with (1.2). In fact, the description of the Levi-Civita connection D of β in the Introduction gives $4R(x,y)z = [[x,y],z]$ for left-invariant vector fields x,y,z, that is, $4R_{ijk}{}^{l} = C_{ij}{}^{p} C_{pk}{}^{l}$. Lemma 2.1(b) now implies our claim, as $T_{ij}^{kl} = -8\beta^{kp} R_{jpi}{}^{l}$ due to (2.4.ii) and (2.6).

3 Proof of Theorem A

We use the component notation of Section 2. According to (1.4) and (1.6),

$$\begin{aligned} (\Lambda\sigma)_{ijkl} &= \Lambda_{ijkl}{}^{rs} \sigma_{rs} \quad \text{with} \quad \Lambda_{ijkl}{}^{rs} = C_{ij}{}^{r} C_{kl}{}^{s} + C_{jk}{}^{r} C_{il}{}^{s} + C_{ki}{}^{r} C_{jl}{}^{s}, \\ (\Pi\zeta)_{pq} &= C^{ij}{}_{p} C^{kl}{}_{q} \zeta_{ijkl}, \quad \text{whenever} \quad \sigma \in [\mathfrak{g}^*]^{\odot 2} \quad \text{and} \quad \zeta \in [\mathfrak{g}^*]^{\wedge 4}. \end{aligned} \tag{3.1}$$

In any real/complex semisimple Lie algebra $\mathfrak{g}$, for $C_{ij}{}^{k}, T_{ij}^{kl}$ as in Section 2,

$$2C^{ij}{}_{p} C^{kl}{}_{q} (C_{ij}{}^{r} C_{kl}{}^{s} + C_{jk}{}^{r} C_{il}{}^{s} + C_{ki}{}^{r} C_{jl}{}^{s}) = 2\delta_p^r \delta_q^s + T_{pq}^{rs} - T_{pq}^{ik} T_{ik}^{rs}. \tag{3.2}$$

In fact, the first of the three terms naturally arising on the left-hand side of (3.2) equals $2\delta_p^r \delta_q^s$ since, by (2.4.i–ii), $C^{ij}{}_{p} C_{ij}{}^{r} = -\delta_p^r$ and $C^{kl}{}_{q} C_{kl}{}^{s} = -\delta_q^s$. The other two terms coincide (as skew-symmetry of $C^{ij}{}_{p}$ in i,j gives $C^{ij}{}_{p} C_{jk}{}^{r} C_{il}{}^{s} = -C^{ij}{}_{p} C_{ik}{}^{r} C_{jl}{}^{s} = C^{ij}{}_{p} C_{ki}{}^{r} C_{jl}{}^{s}$), and so they add up to $4C^{ij}{}_{p} C^{kl}{}_{q} C_{ki}{}^{r} C_{jl}{}^{s}$, that is, $4C^{kl}{}_{q} C_{jl}{}^{s} C_{p}{}^{ji} C_{ik}{}^{r} = 4C^{kl}{}_{q} C^{j}{}_{l}{}^{s} C_{pj}{}^{i} C_{ik}{}^{r} = -4C^{kl}{}_{q} C^{j}{}_{l}{}^{s} (C_{jk}{}^{i} C_{ip}{}^{r} + C_{kp}{}^{i} C_{ij}{}^{r})$; the rightmost equality is due to the Jacobi identity (2.4.iii). The last expression consists of the first term, $-4C^{kl}{}_{q} C^{j}{}_{l}{}^{s} C_{jk}{}^{i} C_{ip}{}^{r} = -4C_{ip}{}^{r} (C^{i}{}_{k}{}^{j} C_{ql}{}^{k} C^{s}{}_{j}{}^{l}) = -4C_{ip}{}^{r} C^{i}{}_{q}{}^{s}$, cf. (2.5), equal, by (2.4.ii) and (2.6), to $C_{pi}{}^{r} C_{q}{}^{si} = T_{pq}^{rs}$, and the second term, $-(2C_{kp}{}^{i} C^{kl}{}_{q})(2C_{ij}{}^{r} C^{j}{}_{l}{}^{s})$, the two parenthesized factors of which are, for the same reasons, nothing else than T_{pq}^{il} and T_{il}^{rs}. This proves (3.2).

Theorem A is now an obvious consequence of (3.1) – (3.2) and Lemma 2.1(b).

4 The spectrum of Ω in real simple Lie algebras

Theorem 4.1 *Let Ω denote the operator with (1.2) corresponding to a fixed real simple Lie algebra $\mathfrak{g}$, and $\Omega^{\mathfrak{h}}$ its analog for $\mathfrak{h}$, chosen so that $\mathfrak{g}$ and $\mathfrak{h}$ satisfy (1.3).*

(i) *Ω is always diagonalizable.*
(ii) *In case (1.3.a), Ω has the same spectrum as $\Omega^{\mathfrak{h}}$, including the multiplicities.*
(iii) *In case (1.3.b), the spectrum of Ω arises from that of $\Omega^{\mathfrak{h}}$ by doubling the original multiplicities and then including 0 as an additional eigenvalue with the required complementary multiplicity. Note that, by Lemma 2.2(c), 0 is not an eigenvalue of $\Omega^{\mathfrak{h}}$.*
(iv) *The eigenspace $\mathrm{Ker}(\Omega - 2\,\mathrm{Id})$ is spanned in case (1.3.a) by β, and in case (1.3.b) by $\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\mathrm{Im}\,\beta^{\mathfrak{h}}$, for the Killing forms β of $\mathfrak{g}$ and $\beta^{\mathfrak{h}}$ of $\mathfrak{h}$.*

Proof By [5, Lemma 14.3(ii) and formulae (14.5) – (14.7)], if $\mathfrak{g}$ is of type (1.3.a), the complexification of $[\mathfrak{g}^*]^{\odot 2}$ may be naturally identified with its (complex) counterpart $[\mathfrak{h}^*]^{\odot 2}$ for $\mathfrak{h}$, in such a way that $\Omega^{\mathfrak{h}}$ and the Killing form $\beta^{\mathfrak{h}}$ become the unique $\mathbb{C}$-linear extensions of Ω and β. Now Lemma 2.2(a)–(b) and (1.5.ii) yield (i), (ii) and (iv) in case (1.3.a).

For $\mathfrak{g}$ of type (1.3.a), Lemma 13.1 of [5] states the following. First, $[\mathfrak{g}^*]^{\odot 2}$ is the direct sum of two Ω-invariant subspaces: one formed by the real parts of $\mathbb{C}$-bilinear symmetric functions $\sigma : \mathfrak{h}\times\mathfrak{h}\to\mathbb{C}$, the other by the real parts of functions $\sigma : \mathfrak{h}\times\mathfrak{h}\to\mathbb{C}$ which are antilinear and Hermitian. Secondly, Ω vanishes on the “Hermitian” summand, and its action on the “symmetric” summand is equivalent, via the isomorphism $\sigma\mapsto\mathrm{Re}\,\sigma$, to the action of $\Omega^{\mathfrak{h}}$ on $\mathbb{C}$-bilinear symmetric functions σ. With diagonalizability of $\Omega^{\mathfrak{h}}$ again provided by Lemma 2.2(a), this proves our remaining claims. (The multiplicities are doubled since the original complex eigenspaces are viewed as real, while the eigenspace $\Omega^{\mathfrak{h}}$ for the eigenvalue 2 consists, by (v) and (1.5.ii), of complex multiples of $\beta^{\mathfrak{h}}$, the real parts of which are precisely the real linear combinations of $\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\mathrm{Im}\,\beta^{\mathfrak{h}}$.) □

Remark 4.2 It is well known [9, p. 30] that, up to isomorphisms, $\mathfrak{sl}(n,\mathbb{R})$ as well as $\mathfrak{su}(p,q)$ with $p+q=n$ and, if n is even, $\mathfrak{sl}(n/2,\mathbb{H})$, are the only real forms of $\mathfrak{sl}(n,\mathbb{C})$.

Lemma 4.3 *The only complex, or real, simple Lie algebras of dimensions less than 7 are, up to isomorphisms, $\mathfrak{sl}(2,\mathbb{C})$ or, respectively, $\mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{su}(2)$, $\mathfrak{su}(1,1)$ and $\mathfrak{sl}(2,\mathbb{C})$, the last one being both complex three-dimensional and real six-dimensional. Consequently,*

(i) *a complex simple Lie algebra cannot be six-dimensional,*
(ii) *there is just one isomorphism type of a complex or, respectively, real simple Lie algebra of dimension 3 or, respectively, 6, both represented by $\mathfrak{sl}(2,\mathbb{C})$,*
(iii) *$\dim\mathfrak{g}\notin\{1,2,4,5\}$ for every real or complex simple Lie algebra $\mathfrak{g}$.*

Proof According to Remark 2.3, in the complex case, only $\mathfrak{sl}(2,\mathbb{C})$ is possible. For real Lie algebras, one can use Remark 4.2 and (1.3). □

Remark 4.4 We can now justify the claim, made in [5, Remark 12.3], that 1 is not an eigenvalue of Ω in any real or complex simple Lie algebra except the ones isomorphic to $\mathfrak{sl}(n,\mathbb{R})$, $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{su}(p,q)$ or, for even n only, $\mathfrak{sl}(n/2,\mathbb{H})$, where $n=p+q\ge 3$.

In fact, by Theorem 2.2 and parts (ii) – (iii) of Theorem 4.1, the only real or complex simple Lie algebras in which Ω has the eigenvalue 1 are, up to isomorphisms, $\mathfrak{sl}(n,\mathbb{C})$ for $n\ge 3$ and their real forms. According to Remark 4.2, these are all listed in the last paragraph.

5 Proof of Theorem B

Let $\sigma \in [\mathfrak{g}^*]^{\odot 2}$ and $\Lambda\sigma = 0$. Consequently, by (1.4), $\sigma([x,y],[z,z']) + \sigma([y,z],[x,z']) + \sigma([z,x],[y,z']) = 0$ for all x,y,z,z' in $\mathfrak{g}$. Thus, $\sigma([x,y],[z,z']) = 0$ whenever $x,y \in \mathfrak{h}_i$ and $z,z' \in \mathfrak{h}_j$ with $j \neq i$. The summands $\mathfrak{h}_i$ and $\mathfrak{h}_j$, being simple, are spanned by such brackets $[x,y]$ and $[z,z']$, and so $\mathfrak{h}_i$ is σ-orthogonal to $\mathfrak{h}_j$. As this is the case for any two summands, we obtain (i), the right-to-left inclusion being obvious. Next,

$$\operatorname{Ker}(\Omega - 2\,\mathrm{Id}) \subset \operatorname{Ker}\Lambda \subset \operatorname{Ker}(\Omega - 2\,\mathrm{Id}) \oplus \operatorname{Ker}(\Omega + \mathrm{Id}). \tag{5.1}$$

In fact, the second inclusion is obvious from Theorem A; the first, from Theorem 4.1(iv), (1.5.i) and (2.1.iii) applied to complex multiples σ of $\beta^{\mathfrak{h}}$.

Part (ii) of Theorem B is immediate, as $[\mathfrak{g}^*]^{\wedge 4} = \{0\}$ when $\dim\mathfrak{g} = 3$. Also, if $\mathfrak{g}$ is simple and $\dim\mathfrak{g} = 6$, Lemma 4.3(i)-(ii) implies that $\mathfrak{g}$ is real and isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. From (2.1.iii), with $\Lambda^{\mathfrak{h}}\sigma = 0$ by (ii), we now get $\mathscr{F} \subset \operatorname{Ker}\Lambda$ for $\mathscr{F} = \{\operatorname{Re}\sigma : \sigma \in [\mathfrak{g}^*]^{\odot 2}\}$, where $[\mathfrak{g}^*]^{\odot 2}$ denotes the space of all symmetric $\mathbb{C}$-bilinear forms $\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$. As $\operatorname{Re}\sigma$ uniquely determines such σ, that is, the operator $\sigma \mapsto \operatorname{Re}\sigma$ is injective, we thus have $\dim_{\mathbb{R}}\mathscr{F} = 12$. The second inclusion in (5.1) is therefore an equality, and $\mathscr{F} = \operatorname{Ker}\Lambda$, for dimensional reasons: $\operatorname{Ker}\Lambda$ contains the subspace $\mathscr{F}$ of real dimension 12, equal, in view of part (a) of Theorem 2.2 and Theorem 4.1(iii), to $\dim_{\mathbb{R}}[\operatorname{Ker}(\Omega - 2\,\mathrm{Id}) \oplus \operatorname{Ker}(\Omega + \mathrm{Id})]$. This yields assertion (iii) in Theorem B.

Let $\mathfrak{g}$ now be simple, with $\dim\mathfrak{g} \notin \{3,6\}$. Due to Theorems 2.2 and 4.1(ii)-(iii), -1 is not an eigenvalue of Ω. Thus, $\operatorname{Ker}(\Omega + \mathrm{Id}) = \{0\}$, and the inclusions in (5.1) are equalities. In view of Theorem 4.1(iv), this completes the proof.

6 Some needed facts from linear algebra

In this section $\mathfrak{g}$ is the underlying real space of a finite-dimensional complex vector space $\mathfrak{h}$ and $J : \mathfrak{g} \to \mathfrak{g}$ is the operator of multiplication by i, also referred to as the *complex structure*. We denote by $\beta^{\mathfrak{h}}$ a fixed nondegenerate $\mathbb{C}$-bilinear symmetric form on $\mathfrak{h}$, so that the $\mathbb{R}$-bilinear symmetric form $\beta = 2\operatorname{Re}\beta^{\mathfrak{h}}$ on $\mathfrak{g}$ is nondegenerate as well. The same applies to any nonzero complex multiple of $\beta^{\mathfrak{h}}$. Thus, β and $\gamma = 2\operatorname{Im}\beta^{\mathfrak{h}}$ constitute a basis of a real vector space $\mathscr{P}$ of $\mathbb{R}$-bilinear symmetric forms on $\mathfrak{g}$. All nonzero elements of $\mathscr{P}$ are nondegenerate. As $\beta^{\mathfrak{h}}$ is $\mathbb{C}$-bilinear, $\gamma(x,y) = -\beta(x,Jy)$ for all $x,y \in \mathfrak{g}$. We use components relative to a basis of $\mathfrak{g}$, as in Section 2.

Lemma 6.1 *The real spaces $\mathfrak{g}$ and $\mathscr{P}$ uniquely determine the pair $(J,\beta^{\mathfrak{h}})$ up to its replacement by $(J, a\beta^{\mathfrak{h}})$ or $(-J, \overline{a\beta^{\mathfrak{h}}})$, with any $a \in \mathbb{C} \smallsetminus \{0\}$.*

Proof For any basis κ,λ of $\mathscr{P}$, replacing $\beta^{\mathfrak{h}}$ by a complex multiple, which leaves $\mathscr{P}$ unchanged, we assume that $\kappa = \beta$. Thus, $\lambda = u\beta + v\gamma$, where $u,v \in \mathbb{R}$ and $v \neq 0$. Writing the equality $\gamma = -\beta(\,\cdot\,,J\,\cdot\,)$ as $\gamma_{rq} = -\beta_{rs}J_q^s$, and then using the reciprocal components $\kappa^{pr} = \beta^{pr}$, we obtain $\kappa^{pr}\lambda_{rq} = \beta^{pr}(u\beta_{rq} - v\beta_{rs}J_q^s) = u\delta_q^p - vJ_q^p$. Now $\pm J$ may be defined by declaring the matrix J_q^p to be the traceless part of $\kappa^{pr}\lambda_{rq}$, normalized so that $J^2 = -\mathrm{Id}$.

At the same time, fixing any $\kappa \in \mathscr{P} \smallsetminus \{0\}$ we may assume, as before, that $\kappa = \beta$. Then κ and $\gamma = -\kappa(\,\cdot\,,J\,\cdot\,)$, determine $2\beta^{\mathfrak{h}}$, being its real and imaginary parts. Combined with the last sentence of the preceding paragraph, this completes the proof. $\square$

The next fact concerns two mappings, $\mathrm{rec}: \mathscr{P} \smallsetminus \{0\} \to \mathfrak{g}^{\odot 2}$ and $\mathfrak{g}^{\odot 2} \ni \mu \mapsto \mu_\flat \in \mathrm{End}\,\mathfrak{g}$. The former sends every nonzero element of $\mathscr{P}$ (which, as we know, is nondegenerate) to its reciprocal. The latter is the operator of index-lowering via β, and takes values in the space of $\mathrm{I\!R}$-linear endomorphisms of $\mathfrak{g}$, which include complex multiples of Id. We then have

$$\{[\mathrm{rec}(\sigma)]_\flat : \sigma \in \mathscr{P} \smallsetminus \{0\}\} = \{a\mathrm{Id} : a \in \mathbb{C} \smallsetminus \{0\}\}. \tag{6.1}$$

Namely, under index raising with the aid of β, the operators $A = a\mathrm{Id}$, for $a \in \mathbb{C} \smallsetminus \{0\}$, correspond to elements μ of $\mathfrak{g}^{\odot 2}$ characterized by $\mu^{pq} = \beta^{pr}A_r^q$. Every such μ is in turn the reciprocal of $\sigma \in [\mathfrak{g}^*]^{\odot 2}$ defined by $\sigma_{pq} = H_p^k\beta_{kq}$, where $H = A^{-1}$ (as $\sigma_{pq}\mu^{sq} = H_p^k\beta_{ks}\beta^{sr}A_r^q = H_p^rA_r^q = \delta_p^q$). Symmetry of μ, and hence σ, is obvious from β-self-adjointness of A. The inverses H of our operators $A = a\mathrm{Id}$ range over nonzero complex multiples of Id as well, and so the resulting symmetric forms σ act on $x,y \in \mathfrak{g}$ by $\sigma(x,y) = \beta(ux + vJx, y)$, where (u,v) range over $\mathrm{I\!R} \smallsetminus \{0\}$. Therefore $\sigma = u\beta - v\gamma$, as required.

Remark 6.2 The relation $\gamma = -\beta(\,\cdot\,, J\,\cdot\,)$ for $\beta = 2\,\mathrm{Re}\,\beta^{\mathfrak{h}}$ and $\gamma = 2\,\mathrm{Im}\,\beta^{\mathfrak{h}}$ shows that, once J is fixed, $\mathrm{Re}\,\beta^{\mathfrak{h}}$ uniquely determines $\beta^{\mathfrak{h}}$. Similarly, $\mathrm{Re}\,C^{\mathfrak{h}}$ and J determine the Cartan three-form $C^{\mathfrak{h}}$ of a complex Lie algebra $\mathfrak{h}$, cf. (1.7). In fact, $\mathrm{Im}\,C^{\mathfrak{h}} = -\mathrm{Re}\,C^{\mathfrak{h}}(\,\cdot\,,\,\cdot\,, J\,\cdot\,)$.

Remark 6.3 The bracket $[\,,\,]$ of a real/complex semisimple Lie algebra is uniquely determined by C and β via (1.7). Knowing C and the set of nonzero scalar multiples of β, rather than β itself, makes $[\,,\,]$ unique up to multiplications by cubic roots of 1. Such factors must be allowed as multiplying $[\,,\,]$ by a scalar r replaces β and C with $r^2\beta$ and r^3C.

Remark 6.4 In the first sentence of Remark 6.3, treating C and β formally, we see that in the complex case $\bar{C}$ and $\bar{\beta}$ determine, via (1.7), the same bracket $[\,,\,]$ as C and β.

7 Proof of Theorem C

For a real/complex Lie algebra $\mathfrak{g}$, let the mapping $\Phi : [\mathfrak{g}^*]^{\wedge 3} \times \mathfrak{g}^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ be defined by $[\Phi(C,\mu)](x,y,z,z') = \mu(C(x,y),C(z,z')) + \mu(C(y,z),C(x,z')) + \mu(C(z,x),C(y,z'))$, where $\mu \in \mathfrak{g}^{\odot 2}$ is treated as a symmetric real/complex-bilinear form on $\mathfrak{g}^*$, and $C(x,y)$ stands for the element $C(x,y,\,\cdot\,)$ of $\mathfrak{g}^*$. If $\mathfrak{g}$ is also semisimple, the isomorphic identification $\mathfrak{g} \approx \mathfrak{g}^*$ provided by the Killing form β induces an isomorphism $[\mathfrak{g}^*]^{\odot 2} \to \mathfrak{g}^{\odot 2}$, which we write as $\sigma \mapsto \sigma^\sharp$. Then, in view of (1.4) and (1.7),

$$\Phi(C, \sigma^\sharp) = \Lambda\sigma \quad \text{for any } \sigma \in \mathfrak{g}^{\odot 2} \text{ and the Cartan three-form } C. \tag{7.1}$$

Theorem C is a trivial consequence of the following result combined with Lemma 4.3(ii) and the fact that, by multiplying a Lie-algebra bracket operation $[\,,\,]$ by a nonzero scalar, one obtains a Lie-algebra structure isomorphic to the original one.

Lemma 7.1 *In a real or complex semisimple Lie algebra $\mathfrak{g}$, the Cartan three-form and the vector-space structure of $\mathfrak{g}$ uniquely determine each of the following objects.*

(a) *The vector subspaces constituting the simple direct summand ideals of $\mathfrak{g}$.*
(b) *Up to a sign, in the real case, the complex structure, defined as in Section 6, of every summand ideal $\mathfrak{g}'$ with $\dim_{\mathrm{I\!R}}\mathfrak{g}' \neq 6$ which is a complex Lie algebra, treated as real.*
(c) *Up to multiplications by cubic roots of 1, the restrictions of the Lie-algebra bracket of $\mathfrak{g}$ to all such summands of dimensions other than 3 or 6.*

(d) *The Lie-algebra isomorphism types of all summand ideals* $\mathfrak{g}'$ *with* $\dim_{\mathrm{I\!R}}\mathfrak{g}' \neq 3$.

Proof Let C be the Cartan three-form of $\mathfrak{g}$. By (7.1), $\operatorname{Ker}\Delta = \{\sigma^\sharp : \sigma \in \operatorname{Ker}\Lambda\}$ for the real/complex-linear operator $\Delta : \mathfrak{g}^{\odot 2} \to [\mathfrak{g}^*]^{\wedge 4}$ given by $\Delta\mu = \Phi(C,\mu)$. Then, if one views all $\mu \in \operatorname{Ker}\Delta \subset \mathfrak{g}^{\odot 2}$ as linear operators $\mu : \mathfrak{g}^* \to \mathfrak{g}$,

(e) *the simple direct summands of* $\mathfrak{g}$ *are precisely the minimal elements, in the sense of inclusion, of the set* $\mathbf{S} = \{\mu(\mathfrak{g}^*) : \mu \in \operatorname{Ker}\Delta,$ and $\dim\mu(\mathfrak{g}^*) = 3$ or $\dim\mu(\mathfrak{g}^*) \geq 6\}$.

In fact, $\mathbf{S}$ consists of the images of those linear endomorphisms $\Sigma : \mathfrak{g} \to \mathfrak{g}$ which correspond via (1.2.b) to elements σ of $\operatorname{Ker}\Lambda$, and have $\operatorname{rank}\Sigma \notin \{0,1,2,4,5\}$. To describe all such Σ, we use the four parts of Theorem B, referring to them as (i) – (iv). Specifically, by (i), our Σ are direct sums of linear endomorphisms Σ_i of the simple direct summands $\mathfrak{g}_i$ of $\mathfrak{g}$, while Σ_i are themselves subject to just two restrictions: one due to the exclusion of ranks $0,1,2,4$ and 5, the other depending, in view of (ii) – (iv), on $d_i = \dim\mathfrak{g}_i$, as follows. If $d_i = 3$, (ii) states that Σ_i is only required to be β-self-adjoint (to reflect symmetry of σ_i related to Σ_i as in (1.2.b)). Similarly, it is clear from (iv) and (2.2) that, with a specific the scalar field $\mathrm{I\!F}$,

$$\Sigma_i \text{ is a nonzero } \mathrm{I\!F}\text{-multiple of } \mathrm{Id} \text{ when } d_i \notin \{3,6\}, \text{ where } \mathrm{I\!F} = \mathbb{C} \text{ if } \mathfrak{g}_i \text{ is either complex or real of type (1.3.b), and } \mathrm{I\!F} = \mathrm{I\!R} \text{ for real } \mathfrak{g}_i \text{ of type (1.3.a).} \tag{7.2}$$

In the remaining case, $d_i = 6$. Then, by (iii) and Theorem B(iii), Σ_i is complex-linear and β-self-adjoint, cf. (2.3) and (2.1.i), but otherwise arbitrary.

The image $\Sigma(\mathfrak{g})$ of any Σ as above is the direct sum of the images of its summands Σ_i, and so it can be minimal only if there exists just one i with $\Sigma_i \neq 0$. For this i, minimality of $\Sigma(\mathfrak{g}) = \Sigma_i(\mathfrak{g}_i)$ implies that $\Sigma(\mathfrak{g}) = \mathfrak{g}_i$. In fact, in view of the last paragraph, the cases $d_i = 3$ and $d_i \notin \{3,6\}$ are obvious (the former since $\operatorname{rank}\Sigma_i \geq 3$) while, if $d_i = 6$, complex-linearity of Σ_i precludes not just $0,1,2,4$ and 5, but also 3 from being its real rank.

We thus obtain one of the inclusions claimed in (e): every minimal element of $\mathbf{S}$ equals some summand $\mathfrak{g}_i$. Conversely, any fixed summand $\mathfrak{g}_i$ is an element of $\mathbf{S}$, realized by Σ with $\Sigma_i = \mathrm{Id}$ and $\Sigma_j = 0$ for all $j \neq i$, cf. Lemma 4.3(iii). Minimality of $\mathfrak{g}_i$ is in turn obvious from (7.2) if $d_i \notin \{3,6\}$, while for $d_i = 3$ or $d_i = 6$ it follows from the restriction on $\operatorname{rank}\Sigma$ combined, in the latter case, with complex-linearity of Σ_i. This yields (e).

Now (a) is obvious from (e), as Δ and $\mathbf{S}$ depend only on C and the vector-space structure of $\mathfrak{g}$. To prove (b) – (c), we fix i with $d_i \notin \{3,6\}$. Elements μ of $\operatorname{Ker}\Delta$ having $\mu(\mathfrak{g}^*) = \mathfrak{g}_i$ correspond, via (1.2.b) followed by the assignment $\sigma \mapsto \mu = \sigma^\sharp$, to endomorphisms Σ of $\mathfrak{g}$ which satisfy (7.2) and vanish on $\mathfrak{g}_j$ for $j \neq i$. Any such μ, now viewed as a bilinear form on $\mathfrak{g}^*$, is therefore obtained from a bilinear form μ_i on $\mathfrak{g}_i^*$ by the trivial extension to $\mathfrak{g}^*$, that is, pullback under the obvious restriction operator $\mathfrak{g}^* \to \mathfrak{g}_i^*$.

If $\mathrm{I\!F} = \mathrm{I\!R}$, it is immediate from (7.2) that the resulting forms μ_i are nonzero multiples of the reciprocal of the Killing form of $\mathfrak{g}_i$, and Remark 6.3 implies (c). Next, let $\mathrm{I\!F} = \mathbb{C}$. We denote $\mathfrak{g}_i$ treated as a complex Lie algebra by $\mathfrak{h}$ and Cartan three-form of $\mathfrak{h}$ by $C^{\mathfrak{h}}$. Formula (6.1) states that, in view of (7.2), the reciprocals of our μ_i are precisely the nonzero elements of the space $\mathscr{P}$ defined in Section 6. Thus, Lemma 6.1, (2.1.ii) and Remark 6.2 imply that C determines the triple $(J,\beta^{\mathfrak{h}},C^{\mathfrak{h}})$ uniquely up to replacements by $(J,a\beta^{\mathfrak{h}},aC^{\mathfrak{h}})$ or $(-J,\overline{a\beta^{\mathfrak{h}}},\overline{aC^{\mathfrak{h}}})$, with $a \in \mathbb{C} \smallsetminus \{0\}$. This proves (b), while using Remarks 6.3 – 6.4 we obtain (c) for $\mathrm{I\!F} = \mathbb{C}$ as well. Finally, (c) and Lemma 4.3(i)–(iii) easily yield (d). □

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Appendix: Meyberg's theorem

For any complex simple Lie algebra $\mathfrak{g}$, the operator Ω with (1.2) is diagonalizable. Its systems $\mathrm{Spec}[\mathfrak{g}]$ of eigenvalues and $\mathrm{Mult}[\mathfrak{g}]$ of the corresponding multiplicities are

$$\begin{aligned}
&\mathrm{Spec}[\mathfrak{sl}_n] = (2,1,2/n,-2/n) \text{ and}\\
&\mathrm{Mult}[\mathfrak{sl}_n] = (1,n^2-1,n^2(n-3)(n+1)/4,n^2(n+3)(n-1)/4), \text{ if } n \geq 4.\\
&\mathrm{Spec}[\mathfrak{sp}_n] = (2,(n+4)/(n+2),-4/(n+2),2/(n+2)) \text{ for even } n \geq 4, \text{ and}\\
&\mathrm{Mult}[\mathfrak{sp}_n] = (1,(n-2)(n+1)/2,n(n+1)(n+2)(n+3)/24,n(n-1)(n-2)(n+3)/12).\\
&\mathrm{Spec}[\mathfrak{so}_n] = (2,(n-4)/(n-2),4/(n-2),-2/(n-2)) \text{ if } n=7 \text{ or } n \geq 9, \text{ while}\\
&\mathrm{Mult}[\mathfrak{so}_n] = (1,(n+2)(n-1)/2,n(n-1)(n-2)(n-3)/24,n(n+1)(n+2)(n-3)/12)
\end{aligned}$$

and, if $\mathfrak{g}$ is one of the exceptional complex Lie algebras $\mathfrak{sl}_2,\mathfrak{sl}_3,\mathfrak{g}_2,\mathfrak{so}_8,\mathfrak{f}_4,\mathfrak{e}_6,\mathfrak{e}_6,\mathfrak{e}_7,\mathfrak{e}_8$,

$$\begin{aligned}
&\mathrm{Spec}[\mathfrak{g}] = (2,(1+w)/6,(1-w)/6), \text{ with } \mathrm{Mult}[\mathfrak{g}] \text{ equal to}\\
&(1,3d[(d+2)w-(d+32)]/[w(11-w)],3d[(d+2)w+(d+32)]/[w(11+w)]),
\end{aligned} \tag{7.3}$$

where $d = \dim\mathfrak{g}$ and $w = [(d+242)/(d+2)]^{1/2}$. This is a result of Meyberg [8] who, instead of our Ω, studied the operator $T = \Omega/2$. (The exponent $1/2$ is missing in [8]). For $\mathfrak{sl}_2$, the resulting "eigenvalue" $4/3$ of multiplicity 0 should be disregarded. All isomorphism types of complex simple Lie algebras are listed above, cf. Remark 2.3.

The dimensions d of $\mathfrak{sl}_2,\mathfrak{sl}_3,\mathfrak{g}_2,\mathfrak{so}_8,\mathfrak{f}_4,\mathfrak{e}_6,\mathfrak{e}_6,\mathfrak{e}_7,\mathfrak{e}_8$ are $3, 8, 14, 28, 52, 78, 133, 248$ [7, p. 21]. The eigenvalues 0 and 1 in (7.3) would correspond to $w=1$ or $w=5$, of which only the latter occurs, for $d=8$ and $\mathfrak{g}=\mathfrak{sl}_3$, in agreement with Lemma 2.2.

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